

Information and Communications Security

WS 2024/25

Assignment 3 – Bonus Material

Cryptography

Fachbereich

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In the following you will find some bonus material to the second exercise including Modular Exponentiation and Fermat's factorization.

Modular Exponentiation

The fast exponentiation algorithm is based on the assumption that for

$x^a \bmod n = (((x^2 \bmod n)^2 \bmod n)^2 \dots)^2 \bmod n$, with $a = 2^k$ (a power of 2)
 squaring is iterated k times. For example, $x^{(2^{2048})}$ can be computed in 2048 modular squaring operations (Knospe 2019, p. 67).

For exponents that are not a power of 2, we can write it as a sum of powers of 2. (Remember that $x^b \cdot x^c = x^{b+c}$)

Example (Lecture Slide 21):

To calculate $7^{17} \bmod 77$ we can write $17 = 2^4 + 2^0$.

We also have $2^4 = 2^2 \cdot 2^2$ and $2^2 = 2^1 \cdot 2^1$.

Powers of 2				
16	8	4	2	1
2 ⁴	2 ³	2 ²	2 ¹	2 ⁰

$$\begin{aligned}
 7^1 &\equiv 7 \bmod 77 \Leftrightarrow 7 \bmod 77 = 7 \\
 7^2 &\equiv 49 \bmod 77 \Leftrightarrow 49 \bmod 77 = 49 \\
 7^4 &\equiv 7^2 \cdot 7^2 \equiv 49 \cdot 49 \bmod 77 \Leftrightarrow 2401 \bmod 77 = 14 \\
 7^8 &\equiv 7^4 \cdot 7^4 \equiv 14 \cdot 14 \bmod 77 \Leftrightarrow 196 \bmod 77 = 42 \\
 7^{16} &\equiv 7^8 \cdot 7^8 \equiv 42 \cdot 42 \bmod 77 \Leftrightarrow 1764 \bmod 77 = 70 \\
 7^{17} &\equiv 7^{16} \cdot 7^1 \equiv 70 \cdot 7 \bmod 77 \Leftrightarrow 490 \bmod 77 = 28
 \end{aligned}$$

Bonus Exercise:

Use modular exponentiation to solve $4^{64} \bmod 7$ and $3^{210} \bmod 15$. You can check your results on <https://sagecell.sagemath.org> with the following syntax (e.g. $3^{128} \bmod 15$):



Type some Sage code below and press Evaluate.

```
1 3**128%15
2
```

Evaluate

Language: Sage

Share

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Help | Powered by SageMath

Fermat's Factorization

Let p, q be prime and $N = pq$. Fermat's factoring represents N as a difference of 2 squares:

$$N = x^2 - y^2 = (x + y)(x - y).$$

First, we start with $x = \lceil \sqrt{N} \rceil$ and then increase x by 1 until $x^2 - N$ is square (so that we can derive y) so that $N = x^2 - y^2$ holds.

This method works because we can represent N as a difference of 2 squares:

$$pq = \left(\frac{1}{2}(p + q)\right)^2 - \left(\frac{1}{2}(p - q)\right)^2 = x^2 - y^2.$$

You will find this explanation with more details in Knospe 2019, p. 178 f.

Bonus Exercise:

Let $N = 247$; then we first set $x \approx \sqrt{N}$. We obtain $x = 16$ and derive $x^2 - N = 9$. Because 9 is square we know that $y = 3$. From above we know that $pq = (x + y)(x - y)$ so we receive $247 = 19 \cdot 13$.

If you need more practice, just choose some primes p, q and derive N . Then use Fermat's factoring to solve N .

Literature

(Knospe 2019) Knospe, Heiko. A Course in Cryptography. Vol. 40. American Mathematical Soc., 2019. <https://ebookcentral.proquest.com/lib/senc/reader.action?docID=5962876>